MAY, 1894.

ANNALS OF MATHEMATICS.

ORMOND STONE, Editor.
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OFFICE OF PUBLICATION: UNIVERSITY OF VIRGINIA.

Volume 8, Number 4.

ALL COMMUNICATIONS should be addressed to ORMOND STONE, University Station, Charlottesville, Va., U. S. A.
Entered at the Post Office as second-class mail matter.

Gift
The University
19 Ja '05

ON THE JACOBIAN ELLIPTIC FUNCTIONS.

By Prof. Irving Stringham, Berkeley, Cal.

1. A Generalized Type of Cyclic Functions.

In a work on Algebra* recently published, I have defined a generalized type of sine and cosine in which a complex base takes the place of e, or natural base. I define them with reference to a modulus, an apparition that subsequently reappears as the modulus of a modified form of the Jacobian elliptic functions, which these generalized cyclic functions serve to define.

With respect to the modulus $\varkappa$, itself defined by the identity

$$\varkappa \quad \varkappa \ (\cos \tfrac{\pi}{2} + i \sin \tfrac{\pi}{2}) ,$$

the definitions of sine and cosine are

$$\sin_\varkappa w \quad \frac{\varkappa}{2} (e^{w/\varkappa} - e^{-w/\varkappa}) ,$$

$$\cos_\varkappa w \quad \frac{1}{2} (e^{w/\varkappa} + e^{-w/\varkappa}) ,$$

in which w is in general complex. The functions $\tan_\varkappa w$, $\cot_\varkappa w$, $\sec_\varkappa w$, $\csc_\varkappa w$, are defined, in the usual way, in terms of $\sin_\varkappa w$ and $\cos_\varkappa w$. These are *modulo-cyclic*, or more briefly, *modo-cyclic* functions.

The analytical theory of these functions is quite as simple as that of either the circular, or the hyperbolic forms of them, which they in fact become, without further reduction, by the substitutions $\varkappa = i = \sqrt{-1}$ and $\varkappa = 1$ respectively; and we may find it advantageous to substitute for the double circular-hyperbolic theory, with its double series of formulæ, the unique modo-cyclic theory and its inclusive table of properties, deriving the former from the latter by the above-mentioned rule of derivation.† For example, it is simpler to remember this rule and write

$$\cos^2_\varkappa w - \varkappa^{-2} \sin^2_\varkappa w = 1 ,$$

$$\sin_\varkappa(w \pm w') = \sin_\varkappa w \cos_\varkappa w' \pm \cos_\varkappa w \cos_\varkappa w' ,$$

$$\cos_\varkappa(w \pm w') = \cos_\varkappa w \cos_\varkappa w' \pm \varkappa^{-2} \sin_\varkappa w \sin_\varkappa w' ,$$

* *Uniplanar Algebra*, p. 101.

† *Uniplanar Algebra*, pp. 102-106.

than to write out the six corresponding circular and hyperbolic formulæ, remembering the variations in algebraic sign for this purpose. The following are the relations, obvious corollaries from the definitions, that connect the various functions with each other:

$$\sin_\varkappa w = \varkappa \sinh \frac{w}{\varkappa}, \qquad \cos_\varkappa w = \cosh \frac{w}{\varkappa},$$

$$\sin_\varkappa w = i\varkappa \sin \frac{w}{i\varkappa}, \qquad \cos_\varkappa w = \cos \frac{w}{i\varkappa},$$

$$\sin_1 w = \sinh w, \qquad \cos_1 w = \cosh w,$$

$$\sin_i w = \sin w, \qquad \cos_i w = \cos w.$$

The derivatives of $\sin_\varkappa$ and $\cos_\varkappa$ are

$$\frac{d \sin_\varkappa w}{dw} = \cos_\varkappa w, \qquad \frac{d \cos_\varkappa w}{dw} = \varkappa^{-2} \sin_\varkappa w.$$

So much is sufficient to properly characterize these modo-cyclic functions and to indicate their place in the list of the ordinary transcendental functions. The more important question of how they behave when made the vehicles of higher analytical processes finds its answer in part in the following brief investigation, which deals primarily with the definitions of the elliptic functions.

2. Rationalization of $dv/\sqrt{a + 2bv + cv^2}$.

As preliminary to the more general discussion, the rationalization of the simpler differential expression $dv/\sqrt{a + 2bv + cv^2}$, by means of the $\sin_\varkappa$-function, is suggestive.

Let

$$V = a + 2bv + cv^2,$$

and suppose this reduced to the form

$$Z = a'(1 + \varkappa^{-2} z^2),$$

by the substitution

$$v = \lambda + \mu z.$$

For this purpose it is sufficient to make $\lambda = -b/c$, whence, by a comparison of coefficients,

$$a' = (ac - b^2)/c,$$

$$\varkappa^{-2} = c^2 \mu^2 / (ac - b^2),$$

in which μ is still arbitrary, so that by properly choosing μ, $\varkappa$ may be made to assume any desired value.

The differential expression then becomes

$$\frac{\mu}{\sqrt{a}} \cdot \frac{dz}{\sqrt{1 + \varkappa^{-2}z^2}},$$

and

$$\frac{dv}{\sqrt{V}} = \frac{1}{\varkappa\sqrt{c}} \cdot \frac{dz}{\sqrt{1 + \varkappa^{-2}z^2}}.$$

The final substitution may now be made in the form

$$z = \sin_\varkappa\left[\varphi + ri\varkappa\frac{\pi}{2}\right],$$

in which r is eventually to be chosen either $= 0$, or $= 1$. The rationalization takes place by virtue of the relation

$$1 + \varkappa^{-2}\sin_\varkappa^2 w = \cos_\varkappa^2 w;$$

and since

$$dz = \cos_\varkappa\left[\varphi + ri\varkappa\frac{\pi}{2}\right]d\varphi,$$

and

$$z = \frac{\varkappa(cv + b)}{\sqrt{ac - b^2}},$$

we have, at once,

$$\frac{dv}{\sqrt{V}} = \frac{d\varphi}{\varkappa\sqrt{c}},$$

or

$$\int\frac{dv}{\sqrt{V}} = \frac{1}{\varkappa\sqrt{c}}\left[\sin_\varkappa^{-1}\frac{\varkappa(cv + b)}{\sqrt{ac - b^2}} - ri\varkappa\frac{\pi}{2}\right].$$

In the ordinary case, when the variables and coefficients are all real, there are four alternatives to be considered, and in each the value of μ is to be so chosen, if possible, as to make the result real.

(i). If c and $ac - b^2$ are both positive, take $r = 0$ and μ such that $\varkappa = 1$; then

$$\int\frac{dv}{\sqrt{V}} = \frac{1}{\sqrt{c}}\sinh^{-1}\frac{cv + b}{\sqrt{ac - b^2}}.$$

(ii). If c and $ac - b^2$ are both negative, take $r = 0$ and μ such that $\varkappa = i$; then

$$\int\frac{dv}{\sqrt{V}} = -\frac{1}{\sqrt{-c}}\sin^{-1}\frac{cv + b}{\sqrt{b^2 - ac}}.$$

(iii). If $c > 0$ and $ac - b^2 < 0$, take $r = 1$ and μ such that $\varkappa = 1$; then, since $\sin_\varkappa\left[\varphi + i\varkappa\frac{\pi}{2}\right] = i\varkappa\cos_\varkappa\varphi$,

$$\int\frac{dv}{\sqrt{V}} = \frac{1}{\sqrt{c}}\cosh^{-1}\frac{cv + b}{\sqrt{b^2 - ac}}.$$

(iv). If $c < 0$ and $ac - b^2 > 0$, take μ such that $z = 1$; then

$$\int_1^v \frac{dv}{\sqrt{V}} = \frac{1}{\sqrt{c}} \left\{ \sinh^{-1} \frac{cv + b}{\sqrt{ac - b^2}} - i\frac{\pi}{2} \right\}.$$

This involves, as it should, an imaginary term, whether $v = 0$, or 1.

3. Reduction of $dv/\sqrt{V}$ to a Normal Form:

$$V = a + 4bv + 6cv^2 + 4dv^3 + ev^4.$$

For the reduction of the elliptic differential $dv/\sqrt{V}$ to a normal form I employ Cayley's linear transformation-theory, first published in the *Cambridge and Dublin Mathematical Journal*, Vol. I (1846), pp. 70–73, reprinted in Cayley's *Collected Mathematical Papers*, Vol. I, pp. 224–227, and also in abridged form in his *Elliptic Functions*, pp. 317–320. Though the first part of the transformation here presented is identical with Cayley's, for completeness' sake so much of the latter is reproduced as is necessary to preserve continuity in the discussion.

Let $(x, y)^4$ denote a quartic in x, y, of the form

$$(x, y)^4 = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

and let this be transformed into

$$(x', y')^4 = a'x'^4 + 4b'x'^3y' + 6c'x'^2y'^2 + 4d'x'y'^3 + e'y'^4,$$

by the linear substitution

$$x = \lambda_1 x' + \mu_1 y', \quad y = \lambda_2 x' + \mu_2 y'.$$

The differential expressions

$$xdy - ydx, \qquad \frac{xdy - ydx}{\sqrt{(x, y)^4}}$$

are hereby transformed into

$$xdy - ydx = (\lambda_1\mu_2 - \lambda_2\mu_1)(x'dy' - y'dx')$$

and

$$\frac{xdy - ydx}{\sqrt{(x, y)^4}} = (\lambda_1\mu_2 - \lambda_2\mu_1) \frac{x'dy' - y'dx'}{\sqrt{(x', y')^4}}.$$

If x, y, x', y' be now replaced by

$$v = y/x, \qquad v' = y'/x',$$

and if

$$V = a + 4bv + 6cv^2 + 4dv^3 + ev^4,$$

$$V' = a' + 4b'v' + 6c'v'^2 + 4d'v'^3 + e'v'^4,$$

$$m = \lambda_1\mu_2 - \lambda_2\mu_1,$$

the differential equation becomes

$$\frac{dv}{\sqrt{V}} = m\frac{dv'}{\sqrt{V'}},$$

and the substitution for this transformation is

$$v = \frac{\lambda_2 + \mu_2 v'}{\lambda_1 + \mu_1 v'}.$$

Since $(x', y')^4$ has been derived from $(x, y)^4$ by the linear transformation

$$x = \lambda_1 x' + \mu_1 y', \quad y = \lambda_2 x' + \mu_2 y',$$

between the invariants

$$g_2 \quad ae - 4bd + 3c^2,$$

$$g_3 \quad ace + 2bcd - ad^2 - eb^2 - c^3,$$

and the corresponding invariants g'_2, g'_3, of $(x', y')^4$, there exist the well-known relations

$$g'_2 = m^4 g_2, \quad g'_3 = m^6 g_3, \quad g'^3_2/g'^2_3 = g^3_2/g^2_3.$$

This much is identical with Cayley's original transformation. In the immediate sequel Cayley's method is continued in the reduction of V' to the form

$$4b'v'(1 + pv')(1 + qv'),$$

instead of the form $a'(1 + pv'^2)(1 + qv'^2)$, and in the determination of the modulus belonging to it.

The reduced form of the differential we seek is one in which V' is of the third degree, and we assume it to be

$$V' \quad 4b'v'(1 + pv')(1 + qv').$$

Comparing the coefficients of this form with those of

$$V' \quad a' + 4b'v' + 6c'v'^2 + 4d'v'^3 + e'v'^4,$$

we find that

$$a' = 0, \qquad c' = \frac{2}{3}b'(p + q),$$

$$e' = 0, \qquad d' = b'pq;$$

whence

$$g'_2 = -4b'^2 pq + \frac{4}{3}b'^2(p + q)^2$$

$$= \frac{4}{3}b'^2(p^2 + q^2 - pq),$$

and

$$g'_3 = 2b' \cdot \frac{2}{3} b' (p+q) \cdot b'pq - \frac{8}{27} b'^3 (p+q)^3$$
$$= \frac{4}{27} b'^3 (p+q)(5pq - 2p^2 - 2q^2) .$$

Hence the equation connecting p and q is

$$\frac{g'^2_3}{g'^3_2} = \frac{(p+q)^2 (5pq - 2p^2 - 2q^2)^2}{4 \cdot 27 \cdot (p^2 + q^2 - pq)^3} = \frac{g_3^2}{g_2^3},$$

involving, other than p and q, only the absolute invariant g_3^2/g_2^3 and numerical coefficients.

This equation will be somewhat simplified by subtracting its two members from unity, becoming thereby

$$1 - \frac{27 g_3^2}{g_2^3} = \frac{4 (p^2 + q^2 - pq)^3 - (p+q)^2 (5pq - 2p^2 - 2q^2)^2}{4 (p^2 + q^2 - pq)^3},$$

an equation which, for the purpose of further simplification, may be expressed wholly in terms of $p - q$ and pq, thus:

$$\frac{\Delta}{g_2^3} = \frac{4 \{(p-q)^2 + pq\}^3 - \{(p-q)^2 + 4pq\} \{pq - 2(p-q)^2\}^2}{4 \{(p-q)^2 + pq\}^3},$$

where

$$\Delta = g_2^3 - 27 g_3^2 ;$$

or, if $p - q = \varphi$ and $pq = \psi$, this may be written

$$\frac{\Delta}{g_2^3} = \frac{4 (\varphi^2 + \psi)^3 - (\varphi^2 + 4\psi)(\psi - 2\varphi^2)^2}{4 (\varphi^2 + \psi)^3}.$$

When the numerator of this fraction is expanded, all the terms except the three that involve $\varphi^2 \psi^2$ mutually cancel one another and the result is

$$\frac{\Delta}{g_2^3} = \frac{27 \varphi^2 \psi^2}{4 (\varphi^2 + \psi)^3} = \frac{27 p^2 q^2 (p-q)^2}{4 (p^2 + q^2 - pq)^3}.$$

This is a reciprocal equation of the sixth degree and has three roots of the form q/p and three of the reciprocal form p/q. In order to its further simplification let

$$R = \frac{27}{4} \cdot \frac{g_2^3}{\Delta},$$

$$\text{I } \theta - 1 = \text{I } q/p - \text{I } p/q ,$$

substitutions that lead at once to a cubic equation in θ from whose roots the values of q/p are easily obtained. In fact, the sextic equation in p, q now is

$$(p^2 + q^2 - pq)^3 - R p^2 q^2 (p-q)^2 = 0 ,$$

and

$$(p-q)^2 = pq(\theta-1),$$

$$p^2+q^2-pq = pq\theta;$$

$$\therefore \quad \theta^3 - R(\theta-1) = 0.$$

This is Cayley's cubic. (See *Elliptic Functions*, p. 319.)

The values of q/p in terms of θ are found to be

$$\frac{q}{p} = \frac{1}{2}\{\theta+1 \pm \sqrt{(\theta-1)(\theta+1)}\};$$

and the two values contained in this formula are reciprocal to each other, a statement easily verified by showing that their product is unity.

For the determination of $m/\sqrt{b'}$, which is a factor in the differential expression $mdv'/\sqrt{V'}$, in terms of p and q, we have the equation

$$g'_2 = \frac{4}{3}b'^2(p^2+q^2-pq) = m^4 g_2,$$

from which we derive

$$\frac{m}{\sqrt{b'}} = \left\{\frac{4}{3}\cdot\frac{p^2+q^2-pq}{g_2}\right\}^{\frac{1}{4}};$$

whence

$$\frac{dv}{\sqrt{V'}} = \left\{\frac{p^2+q^2-pq}{12g_2}\right\}^{\frac{1}{4}} \frac{dv'}{\sqrt{v'(1+pv')(1+qv')}};$$

or, since

$$p^2+q^2-pq = pq\theta,$$

$$\frac{dv}{\sqrt{V'}} = \left\{\frac{pq\theta}{12g_2}\right\}^{\frac{1}{4}} \frac{dv'}{\sqrt{v'(1+pv')(1+qv')}}.$$

The further substitutions

$$qv' = z, \quad q/p = \varkappa^2$$

lead to the following final normal form of the elliptic differential of the first kind:

$$\frac{dv}{\sqrt{V'}} = \left\{\frac{\theta\varkappa^{-2}}{12g_2}\right\}^{\frac{1}{4}} \frac{dz}{\sqrt{z(z+1)(\varkappa^{-2}z+1)}},$$

whose modulus $\varkappa$ is determined by the equations

$$\varkappa = \frac{1}{2}(\sqrt{\theta-1} \pm \sqrt{\theta+3}),$$

$$\theta^3 - \frac{27g_2^3}{4J}(\theta-1) = 0.$$

It is easily shown that the six values of q/p, or $\varkappa^2$, obtained as the solution of the above sextic equation, are the six anharmonic ratios formed with the differences of the roots of the original quartic equation $V = 0$.

4. The Co-efficients of Transformation.

In terms of v the new variable z is

$$z = qv' = q \cdot \frac{\lambda_2 - \lambda_1 v}{\mu_1 v - \mu_2},$$

and for the determination of q we have the equations

$$(\lambda_1\mu_2 - \lambda_2\mu_1)^2 = 4\left[\frac{pq\theta}{12g_2}\right]^{\frac{1}{2}} = 4q\left[\frac{\theta\varkappa^{-2}}{12g_2}\right]^{\frac{1}{2}},$$

$$(\lambda_1\mu_2 - \lambda_2\mu_1) = \frac{\lambda_1^2}{g}(\mu - \lambda),$$

where

$$\lambda = \lambda_2/\lambda_1, \qquad \mu = \mu_2/\mu_1, \qquad g = \lambda_1/\mu_1,$$

while b', as determined from the linear transformation of $(x, y)^4$ into $(x', y')^4$, by comparing coefficients, is

$$b' = \lambda_1^3\mu_1\{a + b(3\lambda + \mu) + 3c(\lambda + \mu)\lambda + d(\lambda + 3\mu)\lambda^2 + e\mu\lambda^3\}.$$

From these equations the value of qq is obtained in the form

$$qq = \frac{\varkappa\lambda_1^4}{4b'g}(\mu - \lambda)^2\sqrt{\frac{12g_2}{\theta}}$$

$$= \frac{\varkappa(\mu - \lambda)^2}{4B}\sqrt{\frac{12g_2}{\theta}},$$

in which

$$B = a + b(3\lambda + \mu) + 3c(\lambda + \mu)\lambda + d(\lambda + 3\mu)\lambda^2 + e\mu\lambda^3.$$

Hence, the final form of z in terms of v and known constants is

$$z = \frac{\varkappa(\mu - \lambda)^2}{4B} \cdot \sqrt{\frac{12g_2}{\theta}} \cdot \frac{\lambda - v}{v - \mu},$$

λ and μ being the two roots of the quartic equation $V = 0$, corresponding to the roots 0 and ∞, of $V' = 0$, regarded as a quartic in v', a remark at once verified by observing that, in the formula of transformation

$$v = \frac{\lambda_2 + \mu_2 v'}{\lambda_1 + \mu_1 v'},$$

the values of v corresponding to $v' = 0$, $v' = \infty$ are respectively

$$v = \lambda_2/\lambda_1 = \lambda, \qquad v = \mu_2/\mu_1 = \mu.$$

Since $-1/p$ and $-1/q$ are the two finite roots of $V' = 0$, the other two roots of $V = 6$ are

$$v = \frac{\lambda gp - \mu}{gp - 1}, \text{ corresponding to } v' = -\frac{1}{p},$$

$$v = \frac{\lambda gq - \mu}{gq - 1}, \text{ corresponding to } v' = -\frac{1}{q},$$

where

$$gp = \frac{(\mu - \lambda)^2}{\varkappa B} \sqrt{\frac{12 g_2}{\theta}},$$

$$gq = \frac{\varkappa (\mu - \lambda)^2}{B} \sqrt{\frac{12 g_2}{\theta}}.$$

5. Definition of the s-Function.

Let the functional relation between z and w be defined by the differential equation

$$\left[\frac{dz}{dw}\right]^2 = z(z+1)(\varkappa^{-2} z + 1),$$

and let $\mathrm{s}w$ be regarded as a solution of this equation, so that by definition

$$z = \mathrm{s}w,$$

$$\left[\frac{d\mathrm{s}w}{dw}\right]^2 = \mathrm{s}w(\mathrm{s}w - 1)(\varkappa^{-2}\mathrm{s}w - 1).$$

Then the differential equation

$$\frac{dv}{\sqrt{V}} = \left[\frac{\theta \varkappa^{-2}}{12 g_2}\right]^{\frac{1}{4}} \frac{dz}{\sqrt{z(z+1)(\varkappa^{-2} z + 1)}}$$

assumes the simpler form

$$\frac{dv}{\sqrt{V}} = \left[\frac{\theta \varkappa^{-2}}{12 g_2}\right]^{\frac{1}{4}} dw,$$

and the rational relation between $\mathrm{s}w$ and v is

$$\mathrm{s}w = \frac{\varkappa (\mu - \lambda)^2}{B} \sqrt{\frac{12 g_2}{\theta}} \cdot \frac{\lambda - v}{v - \mu},$$

and symbolically

$$\int \frac{dv}{\sqrt{V}} = \left[\frac{\theta \varkappa^{-2}}{12 g_2}\right]^{\frac{1}{4}} \mathrm{s}^{-1}\left[\frac{\varkappa (\mu - \lambda)^2}{B} \sqrt{\frac{12 g_2}{\theta}} \cdot \frac{\lambda - v}{v - \mu}\right].$$

6. Definitions of the Functions $\mathrm{sn}_\varkappa$, $\mathrm{cn}_\varkappa$, $\mathrm{dn}_\varkappa$.

By the substitution $z = x^2$ the differential equation

$$\left[\frac{dz}{dw}\right]^2 = z\,(z+1)\,(\varkappa^{-2}z+1)$$

becomes

$$4\left[\frac{dx}{dw}\right]^2 = (x^2+1)\,(\varkappa^{-2}x^2+1),$$

or, if $w = 2u$,

$$\left[\frac{dx}{du}\right]^2 = (x^2+1)\,(\varkappa^{-2}x^2+1):$$

and, by virtue of the previous linear transformation,

$$\frac{dv}{\sqrt{V}} = 2\left[\frac{\theta\varkappa^{-2}}{12g_2}\right]^{\frac{1}{4}} \frac{dx}{\sqrt{(x^2+1)\,(\varkappa^{-2}x^2+1)}} = 2\left[\frac{\theta\varkappa^{-2}}{12g_2}\right]^{\frac{1}{4}} du\,.$$

We may now define outright

$$x = \mathrm{sn}_\varkappa u\,,$$

or we may proceed as follows. Let $x = \sin_\varkappa\varphi$; then

$$dx = \cos_\varkappa\varphi\, d\varphi\,,$$

$$x^2+1 = \sin_\varkappa^2\varphi + 1\,,$$

$$\varkappa^{-2}x^2+1 = \varkappa^{-2}\sin^2\varphi + 1 = \cos_\varkappa^2\varphi\,,$$

and therefore

$$\frac{dx}{\sqrt{(x^2+1)\,(\varkappa^{-2}x^2+1)}} = \frac{d\varphi}{\sqrt{\sin_\varkappa^2\varphi+1}}.$$

If now

$$\frac{d\varphi}{\sqrt{\sin_\varkappa^2\varphi+1}} = du\,,$$

φ is the amplitude of u with respect to the modulus $\varkappa$; symbolically

$$\varphi = \mathrm{am}_\varkappa u\,.$$

With respect to the same modulus, let us also define

$$\sin_\varkappa\varphi = \mathrm{sn}_\varkappa u\,,$$

$$\cos_\varkappa\varphi = \mathrm{cn}_\varkappa u\,,$$

$$\sqrt{\sin_\varkappa^2\varphi+1} = \mathrm{dn}_\varkappa u\,;$$

it then follows that

$$\mathrm{cn}_\varkappa^2 u + \varkappa^{-2}\mathrm{sn}_\varkappa^2 u = 1\,,$$

$$\mathrm{dn}_\varkappa^2 u + \mathrm{sn}_\varkappa^2 u = 1\,.$$

The differentials of these functions obviously are

$$d\mathrm{am}_\kappa u = \mathrm{dn}_\kappa u \,.\, du\,,$$
$$d\mathrm{sn}_\kappa u = \mathrm{cn}_\kappa u \,.\, \mathrm{dn}_\kappa u \,.\, du\,,$$
$$d\mathrm{cn}_\kappa u = \varkappa^{-2}\,\mathrm{sn}_\kappa u \,.\, \mathrm{dn}_\kappa u \,.\, du\,,$$
$$d\mathrm{dn}_\kappa u = \mathrm{sn}_\kappa u \,.\, \mathrm{cn}_\kappa u \,.\, du\,.$$

7. Transition to Cyclo- and Hyperbo-Elliptic Forms.

Since $\mathrm{sin}_\kappa\varphi = i\varkappa \sin(\varphi/i\varkappa)$,

$$\therefore \quad \sqrt{\mathrm{sin}_\kappa^2\varphi + 1} = \sqrt{1 - \varkappa^2 \sin^2(\varphi/i\varkappa)}$$

and

$$\frac{d\varphi}{\sqrt{\mathrm{sin}_\kappa^2\varphi + 1}} = i\varkappa \frac{d(\varphi/i\varkappa)}{\sqrt{1 - \varkappa^2\sin^2(\varphi/i\varkappa)}} = du\,;$$

$$\therefore \quad \frac{\varphi}{i\varkappa} = \mathrm{am}\,\frac{u}{i\varkappa},$$

that is,

$$\mathrm{am}_\kappa u = i\varkappa\,\mathrm{am}\,\frac{u}{i\varkappa}\,.$$

Hence

$$\mathrm{sn}_\kappa u = i\varkappa \sin\frac{\varphi}{i\varkappa} = i\varkappa\,\mathrm{sn}\,\frac{u}{i\varkappa},$$

$$\mathrm{cn}_\kappa u = \cos\frac{\varphi}{i\varkappa} = \mathrm{cn}\,\frac{u}{i\varkappa},$$

and

$$\mathrm{dn}_\kappa u = \sqrt{1 - \varkappa^2\sin^2(\varphi/i\varkappa)} = \mathrm{dn}_\kappa\,\frac{u}{i\varkappa}\,.$$

Similarly, since $\mathrm{sin}_\kappa\varphi = \varkappa \sinh(\varphi/\varkappa)$,

$$\therefore \quad \sqrt{\mathrm{sin}_\kappa^2\varphi + 1} = \sqrt{\varkappa^2\sinh^2(\varphi/\varkappa) + 1}\,,$$

and

$$\frac{d\varphi}{\sqrt{\mathrm{sin}_\kappa^2\varphi + 1}} = \varkappa \frac{d(\varphi/\varkappa)}{\sqrt{\varkappa^2\sinh^2(\varphi/\varkappa) + 1}} = du\,.$$

Hence, if the hyperbolic forms, corresponding to am, sm, cn, dn, be denoted by hm, hs, hc, hd, respectively, we may define

$$\frac{\varphi}{\varkappa} = \mathrm{hm}\,\frac{u}{\varkappa},$$

that is

$$\mathrm{am}_\kappa u = \varkappa\,\mathrm{hm}\,\frac{u}{\varkappa}\,;$$

also

$$\mathrm{sn}_\kappa u = \varkappa \sinh\frac{\varphi}{\varkappa} = \varkappa\,\mathrm{hs}\,\frac{u}{\varkappa},$$

$$\mathrm{cn}_\kappa u = \cosh\frac{\varphi}{\varkappa} = \mathrm{hc}\,\frac{u}{\varkappa},$$

and

$$\mathrm{dn}_\kappa u = \sqrt{\varkappa^2\sinh^2(\varphi/\varkappa) + 1} = \mathrm{hd}\,\frac{u}{\varkappa}\,.$$

Hence also, writing for the moment $w = u/\varkappa$,

$$\operatorname{hm} w = i \operatorname{am} \frac{w}{i},$$

$$i \operatorname{hm} w = \operatorname{am} iw,$$

$$i \operatorname{hs} w = \operatorname{sn} iw,$$

$$\operatorname{hc} w = \operatorname{cn} iw,$$

$$\operatorname{hd} w = \operatorname{dn} iw.$$

I have given a detailed account of these hyperbolic forms in a paper on Hyperbo-Elliptic Functions, which appeared in the *Publications of the Astronomical Society of the Pacific*, Vol. I, pp. 177 *et seq.*

8. Further Development of the Theory.

We may now pass on to further details, and, *pari passu* with the successive steps of the theory as now known, produce the usual formulæ, modified in accordance with the fundamental principles here laid down. For example, we may show that

$$\operatorname{sn}_\varkappa 0 = 0, \quad \operatorname{cn}_\varkappa 0 = 1, \quad \operatorname{dn}_\varkappa 0 = 1,$$

that

$$\operatorname{sn}_\varkappa(-u) = -\operatorname{sn}_\varkappa u, \quad \operatorname{cn}_\varkappa(-u) = \operatorname{cn}_\varkappa u, \quad \operatorname{dn}_\varkappa(-u) = \operatorname{dn}_\varkappa u;$$

that if

$$\int_0^{i\varkappa} \frac{dx}{\sqrt{(x^2+1)(\varkappa^{-2}x^2+1)}} = i\varkappa K,$$

$$\int_0^{i\varkappa'} \frac{dx}{\sqrt{(x^2+1)(\varkappa^{-2}x^2+1)}} = i\varkappa' K', \qquad [\varkappa'^2 = 1-\varkappa^2]$$

then

$$\operatorname{sn}_\varkappa i\varkappa K = i\varkappa, \quad \operatorname{cn}_\varkappa i\varkappa K = 0, \quad \operatorname{dn}_\varkappa i\varkappa K = \varkappa',$$

$$\operatorname{sn}_\varkappa \varkappa(iK + K') = i, \quad \operatorname{cn}_\varkappa \varkappa(iK + K') = \frac{i\varkappa'}{\varkappa}, \quad \operatorname{dn}_\varkappa \varkappa(iK + K') = 0;$$

and that the addition equations for the functions s and $\operatorname{sn}_\varkappa$ are

$$\frac{\mathrm{s}(u+v)}{\mathrm{s}u \cdot \mathrm{s}v} = \left[\frac{\mathrm{s}u - \mathrm{s}v}{\mathrm{s}u\,\mathrm{s}'v - \mathrm{s}v\,\mathrm{s}'u}\right]^2,$$

$$\operatorname{sn}_\varkappa(u+v) = \frac{\operatorname{sn}_\varkappa^2 u - \operatorname{sn}_\varkappa^2 v}{\operatorname{sn}_\varkappa u \operatorname{sn}_\varkappa' v - \operatorname{sn}_\varkappa v \operatorname{sn}_\varkappa' u}.$$

etc. etc.

TRANSFORMATION GROUPS.

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We propose in the following article to give an entirely elementary introduction to the theory of Groups of Infinitesimal Transformations, as that theory has been developed in recent years by Prof. Sophus Lie. The theory to which we wish to introduce the reader is of great importance in Differential Equations (for the treatment of which this theory furnishes the only logical ground), and also in Modern Geometry, Modern Algebra, and parts of Dynamics.

Although the arrangement, and a few of the methods, of this article, are new, the matter is virtually all Lie's. It has been collected from various scattering articles in the "Mathematische Annalen," the "Göttingen Nachrichten," the "Archiv for Mathematik og Naturindenskale" of Christiana; and from the writer's notes of lectures delivered by Lie in the winter of 1887. Since this article was planned, Lie has published the last of a series of masterly works on his theory, and it is the chief object of this elementary introduction to arouse the reader's interest in Lie's most important discoveries.

In the following we shall limit ourselves to two variables, unless expressly stated otherwise. Most of the developments, however, can be immediately extended to n variables.

A transformation of the points of the plane is given by a system of equations of the form

$$x_1 = \varphi(x, y), \quad y_1 = \psi(x, y), \tag{1}$$

where φ and ψ are analytical functions of their arguments, in Weierstrass's sense of the term.

We suppose here that the coordinate axes remain unchanged; but every point of general position, (x, y), is carried over to a new position, of which the coordinates are $\varphi(x, y)$ and $\psi(x, y)$.

If we solve (1) in the form

$$x = \Phi(x_1, y_1), \quad y = \Psi(x_1, y_1), \tag{2}$$

we obtain a transformation which would evidently carry the point $\varphi(x, y)$, $\psi(x, y)$ back to its original position (x, y). The transformation (2) is thus said to be *inverse* to the transformation (1).

The successive performance of (2) and (1) will evidently give us a transformation of the form

$$x_1 = x, \quad y_1 = y,$$

which we call the *identical* transformation. This transformation, therefore, really leaves the position of the point (x, y) unchanged.

Suppose now that we have a family of ∞^1 transformations, given by the equations

$$x_1 = \varphi(x, y, a), \quad y_1 = \psi(x, y, a), \tag{3}$$

where a is a parameter, which can assume ∞^1 continuous values. In general, then, it will *not* be the case that the result of performing any two transformations of the family (3) successively on the points of the plane will be equivalent to the result of performing a third transformation of the family (3) on those points. For instance, the equations

$$x_1 = a - x, \quad y_1 = y$$

represent a family of transformations, which do not possess the above peculiarity. For, if

$$x_2 = a_1 - x_1, \quad y_2 = y_1$$

be a second transformation of the family, we find, when we perform the two successively on the point (x, y), that this point assumes a position given by

$$x_2 = a_1 - a + x, \quad y_2 = y.$$

But the transformation given by the last equations does not belong to our original family

$$x_1 = \text{const.} - x, \quad y_1 = y.$$

If now

$$x_1 = \varphi(x, y, a), \quad y_1 = \psi(x, y, a)$$

be any given transformation of the family (3), and if

$$x_2 = \varphi(x_1, y_1, a_1), \quad y_2 = \psi(x_1, y_1, a_1)$$

be a second transformation of that family, then the transformation which results from performing these two successively, evidently has the form

$$x_2 = \varphi\{\varphi(x, y, a), \psi(x, y, a), a_1\}, \quad y_2 = \varphi\{\varphi(x, y, a), \psi(x, y, a), a_1\}.$$

If it happens, now, that the right hand members of these equations have the general forms

$$\varphi\{\varphi(x, y, a), \psi(x, y, a), a_1\} \qquad \varphi\{x, y, \lambda(a, a_1)\},$$

$$\psi\{\varphi(x, y, a), \psi(x, y, a), a_1\} \qquad \psi\{x, y, \lambda(a, a_1)\},$$

where λ is a constant depending on a and a_1 alone, then we say that *the family of ∞^1 transformations* (3) *form a finite continuous group.*

In words expressed, our condition is evidently that *the result of performing successively any two transformations of the family* (3) *on the points of the plane, must be equivalent to the result of performing a third transformation of that family on those points.*

We shall add to the above, in the groups which we shall consider, the further condition that to every transformation with the parameter a of the family (3), we must be able to assign a certain transformation of the family, with the parameter a, such that the latter transformation shall be the *inverse* of the former. Here a is a function of a alone. From this further condition also follows immediately that the family (3) must contain the *identical* transformation.

Since our family (3) contains *one* arbitrary parameter, we call it, under the above conditions, a *group of one member*; or, symbolically, a G_1.

As an example of a G_1, we may take the family of ∞^1 rotations of the plane around a fixed point, which we shall take as the origin of coordinates. The well-known equations to these rotations are

$$\left.\begin{aligned} x_1 &= x\cos a - y\sin a\,, \\ y_1 &= x\sin a + y\cos a\,, \end{aligned}\right\} \tag{4}$$

where a is the angle through which we turn the radii vectores of the points of the plane.

A second transformation of the family (4) is given by the equations

$$x_2 = x_1\cos a_1 - y_1\sin a_1\,,$$
$$y_2 = x_1\sin a_1 + y_1\cos a_1\,.$$

By eliminating x_1 and y_1, we find

$$\begin{aligned} x_2 &= (x\cos a - y\sin a)\cos a_1 - (x\sin a + y\cos a)\sin a_1 \\ &= x(\cos a\cos a_1 - \sin a\sin a_1) - y(\sin a\cos a_1 + \cos a\sin a_1)\,, \end{aligned}$$

or

$$x_2 = x\cos(a + a_1) - y\sin(a + a_1)\,.$$

Similarly,

$$y_2 = x\sin(a + a_1) + y\cos(a + a_1)\,.$$

But the last two equations evidently represent a transformation of our family (4). Hence the two transformations with parameters a and a_1, performed successively upon the points of the plane, produce the same result as the performance of a single transformation of the same family with the parameter $(a + a_1)$. Also the *inverse* of a transformation with the parameter a is evidently the transformation with the parameter $-a$. We further find the *iden-*

tical transformation in our family by putting $a = 0$. Hence, according to our definition, the family (4) forms a group of one member, or a G_1.

If we have a family of transformations in the plane, the equations to which, of the form (3), contain r independent continuous parameters; and if the conditions are fulfilled, that the successive performance of any two transformations of the family is equivalent to the performance of a third transformation of the family, and that to every transformation of the family there is an *inverse* transformation in the family,—we then say that *the family of ∞^r transformations form a finite continuous group of r members, or a G_r.*

It is evident, for example, that the family of ∞^2 translations in the plane, of the form

$$x_1 = x + a\,, \quad y_1 = y + b\,, \tag{5}$$

form a G_2. For if,

$$x_2 = x_1 + a_1\,, \quad y_2 = y_1 + b_1$$

be a second translation of the family (5), then the successive performance of these two translations is equivalent to the performance of

$$x_2 = x + a + a_1\,, \quad y_2 = y + b + b_1\,.$$

Also, the translation with the parameter $-a$, $-b$ is always the *inverse* of the translation with the parameter a, b. Hence the family of ∞^2 translations (5) form a G_2.

Suppose, now, that we have a G_1 of the form

$$x_1 = \varphi(x, y, a)\,, \quad y_1 = \psi(x, y, a)\,. \tag{6}$$

Let us give the parameter a a fixed value e, and suppose that the corresponding transformation, (which we shall designate as (e), for brevity), carries the point of general position p over to the new position p_1. Then, by hypothesis, the inverse transformation in our G_1 (6), to (e), will carry the point p_1 back to p. Let us suppose that the parameter of the latter transformation is e. A transformation, now, of which the parameter is of the form $e + \delta e$, where δe is an infinitesimal quantity, will carry the point p_1 not exactly back to p, but to a position p', which is at an infinitesimal distance from p. If we perform (e) and $(e + \delta e)$ successively, the result will be equivalent to the performance of a third transformation of our G_1: one that will take the point p directly to the position p'. But this must evidently be an *infinitesimal* transformation.

Let us carry this out analytically. The first transformation is represented by

$$x_1 = \varphi(x, y, e)\,, \quad y_1 = \psi(x, y, e)\,;$$

and the second by

$$x' = \varphi(x_1, y_1, e + \delta e)\,, \quad y' = \psi(x_1, y_1, e + \delta e)\,.$$

By eliminating x_1, y_1, we find the transformation which carries p directly to p'. It has the form

$$x' = \varphi\{\varphi(x, y, e), \psi(x, y, e), e + \partial e\}, \quad y' = \psi\{\varphi(x, y, e), \psi(x, y, e), e + \partial e\}.$$

Develop in powers of ∂e, and we find

$$x' = \varphi\{\varphi(x, y, e), \psi(x, y, e), e\} + \frac{\partial\varphi\{\varphi(x, y, e), \psi(x, y, e), e\}}{\partial e}\partial e + \dots,$$

$$y' = \psi\{\varphi(x, y, e), \psi(x, y, e), e\} + \frac{\partial\psi\{\varphi(x, y, e), \psi(x, y, e), e\}}{\partial e}\partial e + \dots.$$

But, since the transformations (e) and $(\bar{e})$ are *inverse*, we have the identities

$$x \quad \varphi\{\varphi(x, y, e), \psi(x, y, e), \bar{e}\},$$

$$y \quad \psi\{\varphi(x, y, e), \psi(x, y, e), \bar{e}\};$$

and our above equations become

$$x' = x + \frac{\partial\varphi\{\varphi(x, y, e), \psi(x, y, e), \bar{e}\}}{\partial e}\partial e + \dots,$$

$$y' = y + \frac{\partial\psi\{\varphi(x, y, e), \psi(x, y, e), \bar{e}\}}{\partial e}\partial e + \dots;$$

and these equations evidently represent an *infinitesimal transformation*.

It is easy to see that the coefficients of ∂e, above, cannot vanish identically; for they may be written

$$\frac{\partial\varphi(x_1, y_1, \bar{e})}{\partial\bar{e}}, \quad \text{and} \quad \frac{\partial\psi(x_1, y_1, \bar{e})}{\partial\bar{e}};$$

and if these quantities were identically zero, that would necessitate the transformation (6) being free of any parameter, which is contrary to hypothesis.

Since $\bar{e}$ depends upon e alone, the equations to the infinitesimal transformation may evidently be written

$$x' = x + \xi(x, y, e)\partial e + \dots,$$

$$y' = y + \eta(x, y, e)\partial e + \dots;$$

and we see that every G_1 in the plane contains at least *one* infinitesimal transformation. In these equations we suppose e to be a mere member; but ∂e is an infinitesimal variable.

If, now, t be a parameter, the general form of an infinitesimal transformation in the plane will be

$$x_1 = x + \xi(x, y)\partial t + \dots,$$

$$y_1 = y + \eta(x, y)\partial t + \dots.$$

If we neglect, as usual, higher powers than the first of the infinitesimal ∂t, we may write the increments of x and y, under this infinitesimal transformation, in the form

$$\partial x = \xi(x, y)\,\partial t\,, \quad \partial y = \eta(x, y)\,\partial t\,.$$

It is evident that this transformation assigns to every point (x, y) a direction through which it is to be moved, given by

$$\frac{\partial y}{\partial x} = \frac{\eta(x, y)}{\xi(x, y)};$$

and also it assigns to (x, y) an infinitesimal distance

$$\sqrt{\partial x^2 + \partial y^2} = \sqrt{\xi^2 + \eta^2}\,.\,\partial t\,,$$

through which it is to be moved.

We can get a clear and fruitful idea of an infinitesimal transformation, if we neglect terms of the second and higher powers of ∂t, and suppose that all the points of the plane are put into motion, simultaneously, by performing on them the infinitesimal transformation an infinite number of times. In this manner, a point (x, y) will assume a simply infinite number of continuous positions, which form a curve. The whole change of positions of the points of the plane, since it is repeated from moment to moment, may be called a *stationary motion*; and it may be compared with the *flow* of the molecules of a compressible fluid.

Let t represent the time, and let us measure it from a fixed point, say $t = 0$; and suppose that at that moment a certain molecule is at the position (x, y). Then, by means of the stationary motion, in the time t the molecule will be at a new position (x_1, y_1). These new coordinates x_1, y_1 are functions of x, y, and t. If t increases by ∂t, then x_1 and y_1 receive the increments

$$\partial x_1 = \xi(x_1, y_1)\,\partial t\,, \quad \partial y_1 = \eta(x_1, y_1)\,\partial t\,;$$

and we may find x_1, y_1 as functions of t by integrating

$$\frac{\partial x_1}{\xi(x_1, y_1)} = \frac{\partial y_1}{\eta(x_1, y_1)} = \partial t\,, \tag{7}$$

with the condition that for $t = 0$, we have also $x_1 = x$ and $y_1 = y$.

The integration will give us equations of the form

$$x_1 = \Phi(x, y, t)\,, \quad y_1 = \Psi(x, y, t)\,; \tag{8}$$

and our kinematic illustration easily shows that these equations represent a G_1. For, if our stationary motion, in the time t, carries the point (or mole-

cule) (x, y) to the position (x_1, y_1) ; and in the time t, carries (x_1, y_1) to (x_2, y_2),— it is evident that in the time $t + t_1$ the point (x, y) will be carried to the position (x_2, y_2). Thus, our transformations (8) show the peculiarity of a G_1.

Lie has proved, by rigid analysis, that every infinitesimal transformation of the plane belongs to one and only one G_1, the finite transformations of which can be obtained by integrating equations of the form (7). These finite transformations can always be arranged into pairs of "inverses." Furthermore he has proved that every G_1 in the plane contains one and, virtually, only one infinitesimal transformation, where we consider two infinitesimal transformations which differ only by a numerical factor as virtually the same.

Similarly a G_r may be shown to contain r independent infinitesimal transformations. The finite transformations of a G_r (or the G_r itself) is said to be *generated* by performing each of the r independent infinitesimal transformations an infinite number of times.

It is easy to see that we may practically get the infinitesimal transformations of a G_r by assigning to the parameters values differing only by infinitesimal quantities from the values which give the identical transformation. As an example, we may put $a = \partial t$ in the G_1 of rotations, since $a = 0$ gives the identical transformation ; and we see that the equations (4) become

$$x_1 = x - y\partial t\,, \quad y_1 = y + x\partial t\,,$$

which is an infinitesimal rotation.

Since our transformations are certain operations, which are independent of the coordinate axes, it is evident that if we have a set of equations which represent a G_1, they will still represent a G_1 if we introduce new variables.

A little consideration will show that the integration of the system (7) must give results of the form

$$\left.\begin{aligned} \Omega(x_1, y_1) &= \Omega(x, y)\,, \\ \mathrm{W}(x_1, y_1) - t &= \mathrm{W}(x, y)\,; \end{aligned}\right\} \tag{9}$$

since, for $t = 0$, we must have $x_1 = x$ and $y_1 = y$. The equations (9) represent our G_1 ; and if we introduce new variables by means of the substitution

$$x_1' = \Omega(x_1, y_1), \quad x' = \Omega(x, y)\,,$$
$$y_1' = \mathrm{W}(x_1, y_1), \quad y' = \mathrm{W}(x, y)\,,$$

the equations to our G_1 take the form

$$x_1' = x'\,,$$
$$y_1' = y' + t\,.$$

But the last equations evidently represent a group of translations. Hence, *any G_1 in the plane, can, by a proper choice of variables, be brought to the form of a translation.*

We shall now derive a very useful symbol to represent an infinitesimal transformation.

If $$x_1 = \varphi(x, y, t), \quad y_1 = \psi(x, y, t),$$ be the finite equations to a G_1, we can evidently consider any function $f(x_1, y_1)$ as a function of x, y, and t; and for the value of t which gives the identical transformation, say for the value $t = 0$, we must have $f(x_1, y_1) = f(x, y)$; and when t varies, $f(x_1, y_1)$ also varies. Thus we are led to ask for the increment δf which this function receives, when x_1 and y_1 receive their respective increments

$$\delta x_1 = \xi(x_1, y_1)\,\delta t, \quad \delta y_1 = \eta(x_1, y_1)\,\delta t.$$

We have

$$\delta f(x_1, y_1) \quad \frac{\partial f(x_1, y_1)}{\partial x_1}\,\delta x_1 + \frac{\partial f(x_1, y_1)}{\partial y_1}\,\delta y_1$$

$$\left\{\xi(x_1, y_1)\frac{\partial f(x_1, y_1)}{\partial x_1} + \eta(x_1, y_1)\frac{\partial f(x_1, y_1)}{\partial y_1}\right\}\delta t.$$

This is the increment of $f(x_1, y_1)$ under the infinitesimal transformation of our G_1. If, now, $t = 0$,

$$\delta f(x, y) \quad \left\{\xi(x, y)\cdot\frac{\partial f(x, y)}{\partial x} + \eta(x, y)\cdot\frac{\partial f(x, y)}{\partial y}\right\}\delta t.$$

It is obvious that if we know what increment a function $f(x, y)$ receives under the infinitesimal transformation of a G_1, we may consider the infinitesimal transformation, or the G_1, *known*. Hence we are led, quite naturally, to introduce the symbol

$$\xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y},$$

to represent the infinitesimal transformation of a G_1. Thus, when we speak of the infinitesimal transformation,

$$-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y},$$

we mean the transformation which assigns to x and y, respectively, the increments

$$\delta x = -y\delta t, \quad dy = x\delta t.$$

We shall represent our above symbol in a still shorter form by writing

$$X(f) \quad \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y}.$$

We see, therefore, that $X(x)$ means, *put x in place of f in the above identity*; and similarly for $X(y)$. We find immediately

$$X(x) \quad \xi, \quad X(y) \quad \eta;$$

so that

$$X(f) \quad X(x)\frac{\partial f}{\partial x} + X(y)\frac{\partial f}{\partial y}.$$

It is easy to see that if we introduce new variables x', y' into this symbol, we find

$$X(f) \quad X(x')\frac{\partial f}{\partial x'} + X(y')\frac{\partial f}{\partial y'}.$$

Let us show, at once, the convenience of our new symbol, which we may write Xf. If we develop $f(x_1, y_1)$ by Maclaurin's formula, we find

$$f(x_1, y_1) \quad f_1 \quad \left\{ f(x_1, y_1) \right\}_{t=0} + \left\{ \frac{df_1}{dt} \right\}_{t=0} \frac{t}{1} + \left\{ \frac{d^2 f_1}{dt^2} \right\}_{t=0} \frac{t^2}{1 \cdot 2} + \cdots.$$

Now

$$\frac{df_1}{dt} \quad \frac{\partial f_1}{\partial x_1}\cdot\frac{dx_1}{dt} + \frac{\partial f_1}{\partial y_1}\cdot\frac{dy_1}{dt}.$$

But

$$\frac{dx_1}{dt} = \xi(x_1, y_1), \quad \frac{dy_1}{dt} = \eta(x_1, y_1),$$

as usual; and, hence,

$$\frac{df_1}{dt} \quad \xi(x_1, y_1)\frac{\partial f}{\partial x_1} + \eta(x_1, y_1)\frac{\partial f}{\partial y_1}.$$

But the last expression is nothing else than our symbol Xf, written in the variables x_1, y_1. Let us call this $X_1 f$. Hence

$$\frac{df_1}{dt} \quad X_1 f.$$

Thus, any function f_1 of x_1 and y_1 gives, when differentiated with respect to t, $X_1 f$. But $X_1 f$ is itself such a function. Hence

$$\frac{d^2 f_1}{dt^2} \quad \frac{d(X_1 f)}{dt} \quad X_1(X_1 f).$$

Similarly,

$$\frac{d^3 f_1}{dt^3} \quad X_1(X_1(X_1 f)),$$

and the law of formation of our coefficients is now obvious.

If we put $t = 0$, then x_1 and y_1 are changed into x and y; $X_1 f$ becomes Xf; $X_1(X_1 f)$ becomes $X(Xf)$; etc. Thus we arrive at the important ex-

pansion

$$f(x_1, y_1) = f(x, y) + \frac{t}{1} Xf + \frac{t^2}{1 \cdot 2} X(Xf) + \dots . \tag{10}$$

This holds, of course, in particular, for $f = x$, and $f = y$. Thus

$$\left.\begin{aligned} x_1 &= x + \frac{t}{1} \cdot X(x) + \frac{t^2}{1 \cdot 2} X(X(x)) + \dots , \\ y_1 &= y + \frac{t}{1} \cdot X(y) + \frac{t^2}{1 \cdot 2} X(X(y)) + \dots ; \end{aligned}\right\} \tag{11}$$

and these are the finite equations of the G_1 of which

$$Xf \quad \xi(x, y) \frac{\partial f}{\partial x} + \eta(x, y) \frac{\partial f}{\partial y}$$

is the infinitesimal transformation.

We see that (11) is only a new form of (9). In fact, (11) is the result of integrating the system (7) in an infinite series, with the condition that for $t = 0$, also $x_1 = x$, $y_1 = y$. Since x_1, y_1 are analytical functions of x, y, and t, it is certain that the series (11) converge for values of t near to $t = 0$.

Suppose now we demand that the function $\Omega(x, y)$ shall be *invariant* under all transformations of a given G_1. That is, analytically,

$$\Omega(x_1, y_1) = \Omega(x, y),$$

where the finite equations of our G_1 have the form

$$x_1 = \varphi(x, y, t), \quad y_1 = \psi(x, y, t),$$

and

$$Xf \quad \xi(x, y) \frac{\partial f}{\partial x} + \eta(x, y) \frac{\partial f}{\partial y}$$

is the infinitesimal transformation. Thus, by (10), we must have

$$\Omega(x, y) + \frac{t}{1} \cdot X(\Omega) + \frac{t^2}{1 \cdot 2} X(X(\Omega)) + \dots = \Omega(x, y),$$

or, as an obviously necessary and sufficient condition,

$$X(\Omega) \quad 0.$$

If this condition is fulfilled, $\Omega(x, y)$ is said to be the *invariant* of the G_1.

Every point (x, y) in the plane describes, when we perform on it the infinitesimal transformation of the G_1 an infinite number of times, a continuous curve, which we will call the *locus of the point under the* G_1. The direction

in which the point (x, y) is moved, is always given by

$$\frac{\delta y}{\delta x} = \frac{\eta(x, y)}{\xi(x, y)}.$$

If now $\Omega(x, y)$ is the *invariant* of the G_1, Ω must satisfy the partial differential equation

$$Xf = \xi \frac{\partial \Omega}{\partial x} + \eta \frac{\partial \Omega}{\partial y} = 0.$$

Hence the integral curves,

$$\Omega(x, y) = \text{const.},$$

will have, in each point, the tangential direction

$$\frac{\delta y}{\delta x} = \frac{\eta(x, y)}{\xi(x, y)}.$$

That is to say, the invariant of a G_1 in the plane, being put equal to a constant, will represent the curves in the plane, which we call the *loci* of the points of the plane under the G_1, or simply the *loci* of the G_1. It is evident that there are ∞^1 of these *loci*, one through every point of general position in the plane.

We see that any point, or points, in the plane for which

$$\xi(x, y) = \eta(x, y) = 0$$

are absolutely invariant under the infinitesimal transformation

$$Xf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}.$$

A *family* of curves in the plane is said to be invariant under a G_1, when, under the transformations of the G_1, *every curve of the family is transformed to a curve of the same family*. The *loci* of a G_1 are evidently an invariant family; and every curve of this family is separately invariant.

If

$$w(x, y) = \text{const.}$$

be any family of curves in the plane, which, as a family, are invariant under a G_1 in the plane; and if

$$w(x_1, y_1) = \text{const.}$$

represents, as it may, the family, after the finite transformations of the G_1 have been performed upon it; then, since the family is invariant, a condition of the form

$$w(x_1, y_1) = \Omega(w(x, y))$$

must hold, by means of the finite transformations of the G_1. Here Ω may be any function of w.

Developing the left hand of the above equation in the usual manner, we find as a necessary and sufficient condition that the family of curves

$$w(x, y) = \text{const.}$$

shall be invariant under the G_1, Xf, is

$$X(w(x, y)) = \Omega(w(x, y)).$$

If a relation of this form holds, we say that the family of curves *admits of* the transformations of the G_1, Xf.

As an example, we see that if

$$Xf \quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

(the G_1 of so-called *similitudinous* transformations), then the family of concentric circles

$$w \quad x^2 + y^2 = \text{const.}$$

is invariant. For we find

$$X(w(x, y)) \quad 2(x^2 + y^2) = 2w(x, y).$$

Thus the criterium holds.

We may mention that one of the most important methods of classifying groups is into *primitive* and *imprimitive*. A group of transformations in the plane, which leaves a family of ∞^1 curves invariant, is said to be *imprimitive*. If no family of ∞^1 curves is invariant, the group is *primitive*.

It is evident that every G_1 in the plane is *imprimitive*, since it always has a family of ∞^1 invariant *loci*.

For other methods of classification, we have to refer to Lie's works.

After discussing thus quite extensively the properties of the G_1 in the plane, we will say a few words about Lie's fundamental theorems for groups of more than one parameter, or member.

If $X_1 f$ and $X_2 f$ are two transformations of a G_r, we know that, by definition, the transformation which is equivalent to the successive performance of $X_1 f$ and $X_2 f$ also belongs to the G_r.

Let us represent by $X_1 f$ the infinitesimal transformation

$$x_1 = x + \xi(x, y)\,\delta t,$$
$$y_1 = y + \eta(x, y)\,\delta t;$$

and by $X_2 f$,

$$x_2 = x_1 + \xi_1(x_1, y_1)\,\delta\tau,$$
$$y_2 = y_1 + \eta_1(x_1, y_1)\,\delta\tau.$$

Here we know that ∂t and $\partial\tau$ are infinitesimals, of which the ratio $\partial t : \partial\tau$ is arbitrary.

Dropping infinitesimals of an order higher than the first, we find

$$x_2 = x + \xi(x, y)\,\partial t + \xi_1(x, y)\,\partial\tau\,,$$
$$y_2 = y + \eta(x, y)\,\partial t + \eta_1(x, y)\,\partial\tau\,;$$

and this new transformation, which we shall call $X_3 f$, must belong to our group.

It is clear that, where λ and μ are certain finite quantities, and $\partial\varepsilon$ an infinitesimal, we may write

$$\partial t = \lambda\,.\,\partial\varepsilon\,, \quad \partial\tau = \mu\,.\,\partial\varepsilon\,,$$

and hence

$$x_2 = x + \{\lambda\,.\,\xi(x, y) + \mu\,.\,\xi_1(x, y)\}\,\partial\varepsilon\,,$$
$$y_2 = y + \{\lambda\,.\,\eta(x, y) + \mu\,.\,\eta_1(x, y)\}\,\partial\varepsilon\,.$$

From the last equations, we see that we may write

$$X_3 f = \lambda\,.\,X_1 f + \mu\,.\,X_2 f\,.$$

Thus, we find that if a G_r contain $X_1 f$ and $X_2 f$, it must also contain $\lambda\,.\,X_1 f + \mu X_2 f$, where the ratio $\lambda : \mu$ is arbitrary.

Similarly, we may show that, universally, if a G_r contain the r infinitesimal transformations $X_1 f$, $X_2 f$, $\ldots$, $X_r f$, it must contain every infinitesimal transformation of the general form,

$$\sum_1^r \lambda_i X_i f\,. \qquad (\lambda_i \text{ constants})$$

It is easy to see the propriety of calling the r infinitesimal transformations *independent*, when no relation of the form

$$\sum_1^r \mu_i X_i f = 0$$

can be established, where the μ_i are constants, not all of which are zero. In a G_r, there are always r *independent* infinitesimal transformations. In the finite equations to the G_r, as we saw, there are always r *independent* parameters.

We shall now develop another method for finding new transformations belonging to a group, of which we know certain transformations. Let S and T represent any two finite transformations of a G_r of the general forms

$$(S)\quad x' = \varphi(x, y)\,, \qquad y' = \psi(x, y)\,;$$
$$(T)\quad u' = \varphi_1(u, v)\,, \qquad v' = \psi_1(u, v)\,.$$

Suppose now that we perform the transformation T *upon* the transformation S. That is, we must introduce new variables into S in place of x', y' and x, y by means of T. This change will be accomplished by means of equations of the form

$$u = \varphi_1(x, y), \qquad v = \psi_1(x, y), \tag{12}$$

$$u' = \varphi_1(x', y'), \qquad v' = \psi_1(x', y'). \tag{13}$$

By solving (12) we find equations of the form

$$x = \Phi_1(u, v), \quad y = \Psi_1(u, v).$$

Hence, from the definition of S,

$$x' = \varphi\{\Phi_1(u, v), \Psi_1(u, v)\},$$

$$y' = \psi\{\Phi_1(u, v), \Psi_1(u, v)\}.$$

Substituting now in (13), we get, finally,

$$\begin{aligned} u' &= \varphi_1\{\varphi[\Phi_1(u, v), \Psi_1(u, v)], \psi[\Phi_1(u, v), \Psi_1(u, v)]\}, \\ v' &= \psi_1\{\varphi[\Phi_1(u, v), \Psi_1(u, v)], \psi[\Phi_1(u, v), \Psi_1(u, v)]\}. \end{aligned} \tag{14}$$

If, as is convenient, we indicate the successive performance of the two transformations S and T, by ST; and call the inverse transformation to T, T^{-1}; we easily see that the transformation (14) is made of the three, T^{-1}, S, and T, performed successively. That is, the transformation (14) may be written $T^{-1}ST$. Since S, T, and T^{-1} belong to our G_r, of course $T^{-1}ST$ is also a transformation of the G_r.

If, now, S and T are two finite transformations of a G_r, and Xf and Yf are the infinitesimal transformations of the G_r which *generate* respectively S and T, we wish to find the infinitesimal transformation, which generates the finite transformation $T^{-1}ST$.

Suppose that our transformations have the following general forms, respectively,

$$(S) \quad \begin{cases} x_1 = x + t\,.\,\xi(x, y) + \dfrac{t^2}{1\,.\,2}X(\xi) + \dots, \\ y_1 = y + t\,.\,\eta(x, y) + \dfrac{t^2}{1\,.\,2}X(\eta) + \dots; \end{cases}$$

$$Xf \quad \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y};$$

$$(T) \quad \begin{cases} x' = x + \tau \cdot \xi(x, y) + \dfrac{\tau^2}{1 \cdot 2} V(\xi) + \dots \\ y' = y + \tau \cdot \eta(x, y) + \dfrac{\tau^2}{1 \cdot 2} V(\eta) + \dots ; \end{cases}$$

$$V f \equiv \xi(x, y) \frac{\partial f}{\partial x} + \eta(x, y) \frac{\partial f}{\partial y} .$$

Now Xf is a transformation in the variables x, y. Suppose that, by means of T, we introduce into Xf the new variables x', y'; that is, we suppose, for the present, that T is a change of axes. Then we know that Xf assumes the form

$$X(x') \frac{\partial f}{\partial x'} + X(y') \frac{\partial f}{\partial y'} .$$

From the defining equations of the transformation T, we see

$$\left.\begin{aligned} X(x') &= X(x) + \tau X(\xi) + \frac{\tau^2}{1 \cdot 2} XV(\xi) + \dots , \\ X(y') &= X(y) + \tau X(\eta) + \frac{\tau^2}{1 \cdot 2} XV(\eta) + \dots ; \end{aligned}\right\} \tag{15}$$

and we wish to express the right hand members of these equations in x' and y'.

Now $\xi(x, y)$ is, by means of T, a function of x', y'. Hence, by an obvious reversing of the general formula (10), we see that we may write

$$X(x) \quad \xi(x, y) = \xi(x', y') - \tau V(\xi(x', y')) + \frac{\tau^2}{1 \cdot 2} VV(\xi) + \dots$$

$$X(y) \quad \eta(x, y) = \eta(x', y') - \tau V(\eta(x', y')) + \frac{\tau^2}{1 \cdot 2} VV(\eta') \dots .$$

In the same way, we have

$$X(\xi(x, y)) = X(\xi(x', y')) - \tau VX(\xi') + \dots ,$$

$$X(\eta(x, y)) = X(\eta(x', y')) - \tau VX(\eta') + \dots ;$$

and

$$XV(\xi(x, y) = XV(\xi(x', y') - \dots$$

$$XV(\eta(x, y) = XV(\eta(x', y')) - \dots$$

We are now ready to find the right hand members of equations (15), in the new variables x', y'; and hence our new transformation

$$X(x') \frac{\partial f}{\partial x'} + X(y') \frac{\partial f}{\partial y'} .$$

By carrying out the substitutions, we find for our new transformation

$$\xi(x', y')\frac{\partial f'}{\partial x'} + \eta(x', y')\frac{\partial f'}{\partial y'} - \tau\left[(X(\xi') - Y(\xi'))\frac{\partial f'}{\partial x'} + (X(\eta') - Y(\eta'))\frac{\partial f'}{\partial y'}\right]$$
$$+ \frac{\tau^2}{1.2}\Big[(YY(\xi') - 2YX(\xi') + XY(\xi'))\frac{\partial f'}{\partial x'}$$
$$+ (YY(\eta') - 2YX(\eta') + XY(\eta'))\frac{\partial f'}{\partial y'}\Big] + \dots.$$

Let us now, for brevity, indicate, with Poisson, by the symbol, (X, Y), the operation

$$X(Yf') - Y(Xf').$$

We then easily see that the coefficient of τ in the above transformation is exactly (X, Y), written in the variables x', y'. A little consideration will also show that, if the expression (X, Y) is itself combined with Yf', just as Xf' and Yf' are in the bracket expression (X, Y), the resulting expression, of the form $((X, Y), Y)$, will be the coefficient of $\frac{\tau^2}{1.2}$.

As nothing depends upon the accented variables, we may drop the accents, and say that the infinitesimal transformation which generates $T^{-1}ST$ has, as far as the first three terms of the series, the general form

$$Xf' + \tau(X, Y) + \frac{\tau^2}{1.2}((X, Y), Y) + \dots.$$

But since Xf' is a transformation of our G_r, we see by subtraction that

$$\tau(X, Y) + \frac{\tau^2}{1.2}((X, Y), Y) + \dots$$

must belong, also, to our G_r. Divide out by τ, and then put $\tau = 0$, and *we see that*

$$(X, Y)$$

must belong to our group, if Xf' and Yf' do.

Suppose now that

$$X_1f', X_2f', \dots, X_rf'$$

are the r independent infinitesimal transformations of our G_r. Then every infinitesimal transformation of the G_r is included in the general form

$$\sum_1^r \lambda_i X_i f'. \qquad (\lambda_i = \text{const.})$$

But if our G_r contain $X_i f$ and $X_k f$, we have just seen, above, that it must also contain (X_i, X_k). *Hence, for all the infinitesimal transformations of our G_r, relations must hold of the general form*

$$(X_i f, X_k f) = \sum_1^r c_{iks} X_s f, \quad (i, k = 1, 2, \ldots, r) \tag{16}$$

the c_{iks} being constants. The equations (16) express analytically the *fundamental theorem* of Lie's works.

It can be shown that, *if r independent infinitesimal transformations satisfy the relations* (16), *they will then generate* (*or form*) *a G_r.* This definition of a G_r can be shown to be identical with that given at the beginning of this article, where we made use of the *finite* equations of the G_r.

For any *three* equations of the form $X_i f$, $X_k f$, and $X_s f$, the celebrated Jacobian identity must hold good. Thus, we must have

$$((X_i, X_k), X_j) + ((X_k, X_j), X_i) + ((X_j, X_i), X_k) = 0. \tag{17}$$

As Engel has remarked, we may verify this identity by merely performing the operations indicated. Thus

$$((X_i, X_k), X_j) = X_i(X_k(X_j)) - X_k(X_i(X_j)) - X_j(X_i(X_k)) + X_j(X_k(X_i)).$$

If we write out $((X_k, X_j), X_i)$ and $((X_j, X_i), X_k)$ in full in a similar manner, we find that the terms of (17) will all cancel in pairs.

We see, thus, that the c_{iks} in (16) are not arbitrary, but must fulfill certain conditions which flow from Jacobi's identity. These constants are called the *setting* of the G_r, and one of the problems of Lie's theory is *to find all groups of a given setting.*

We may remark that Lie has found all groups in one and two variables; and also the most interesting ones in three variables. We hope to give some of the more interesting and elementary of these results in a subsequent paper.

355

REQUIRED the locus of the point in the normal to a conic, which is equally distant from the focus and the foot of the normal. [*Geo. R. Dean.*]

SOLUTION.

Taking the origin at the centre of the ellipse we have for the equation of the normal

$$y - b\sin\varphi = \frac{a}{b}\tan\varphi\,(x - a\cos\varphi)$$

or

$$y = \frac{a}{b}\tan\varphi\,.\,x - (a-b)\sin\varphi\,. \qquad (1)$$

The locus required is the locus of the intersection of this normal with the line,

$$y = -\frac{a}{b}\tan\varphi\,.\,(x - ae)\,, \qquad (2)$$

which passes through the focus. From (2)

$$\tan\varphi = -\frac{by}{a\,(x-ae)} \quad \text{and} \quad \sin\varphi = -\frac{by}{\sqrt{b^2y^2 + a^2\,(x-ae)^2}}.$$

Eliminating φ from (1) by means of these equations, we have, after reduction,

$$[(2x-ae)^2 - a^2e^4]\,(x-ae)^2 + (1-e^2)\,y^2\,(2x-ae)^2 = 0\,,$$

a curve having $(2x-ae)^2 = 0$ for the equation of two coincident asymptotes. The curve cuts the axis of abscissæ at the points $x = \frac{1}{2}ae\,(1 \pm e)$, and the focus is a conjugate point. [*E. S. Crawley.*]

357

GIVEN $\tan_\kappa w = x + iy$, wherein

$$\tan_\kappa w = \varkappa\,(e^{w/\kappa} - e^{-w/\kappa})/(e^{w/\kappa} + e^{-w/\kappa})\,,$$

$$w = u + iv, \quad i = \sqrt{-1}\,,$$

and

$$\varkappa = m\,(\cos\vartheta + i\sin\vartheta)\,;$$

determine u and v as real quantities in terms of x, y, m, and ϑ.

[*Irving Stringham.*]

SOLUTION.

Let

$$\frac{e^{w/\kappa} - e^{-w/\kappa}}{e^{w/\kappa} + e^{-w/\kappa}} = \frac{x + iy}{m\cos\vartheta + im\sin\vartheta};$$

$$\therefore \quad e^{2w/\kappa} = \frac{m\cos\vartheta + x + i(m\sin\vartheta + y)}{m\cos\vartheta - x + i(m\cos\vartheta - y)} = \frac{s \cdot e^{i\phi}}{t \cdot e^{i\psi}} = \frac{s}{t} e^{i(\phi - \psi)},$$

in which s is the tensor of the numerator and t of the denominator; whence

$$w = \frac{1}{2}\kappa\left[\log_e \frac{s}{t} + (\varphi - \varphi' + 2h\pi)\,i\right],$$

in which h is an arbitrary integer;

$$\therefore \quad u = \frac{1}{2} m \cdot \cos\vartheta \cdot \log \frac{s}{t} - \frac{1}{2} m \cdot \sin\vartheta \cdot (\varphi - \varphi' + 2h\pi),$$

$$v = \frac{1}{2} m \cdot \cos\vartheta \cdot (\varphi - \varphi' + 2h\pi) + \frac{1}{2} m \cdot \sin\vartheta \cdot \log \frac{s}{t};$$

in which

$$\log \frac{s}{t} = \frac{1}{2} \log \frac{m^2 + x^2 + y^2 + 2mx\cos\vartheta + 2my\sin\vartheta}{m^2 + x^2 + y^2 - 2mx\cos\vartheta - 2my\sin\vartheta},$$

$$\varphi = \tan^{-1} \frac{m\sin\vartheta + y}{m\cos\vartheta + x}, \qquad \varphi' = \tan^{-1} \frac{m\sin\vartheta - y}{m\cos\vartheta - x};$$

and, consequently,

$$\varphi - \varphi' = \tan^{-1} \frac{2m(y\cos\vartheta - x\sin\vartheta)}{m^2 - x^2 - y^2}.$$

If

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1} \frac{y}{x},$$

$$\lg \frac{s}{t} = \frac{1}{2} \lg \left\{\frac{m^2 + r^2 + 2mr\cos(\theta - \vartheta)}{m^2 + r^2 - 2mr\cos(\theta - \vartheta)}\right\};$$

$$\varphi - \varphi' = \tan^{-1} \frac{2mr\sin(\theta - \vartheta)}{m^2 - r^2}.$$

[*J. B. Shaw, Jr.*]

358

A HEMISPHERICAL bowl, external and internal radii R and r, has a sphere of the same material, radius a, suspended from the edge of the rim. Determine the inclination of the plane of the rim to the horizontal plane on which the bowl stands. [*Artemas Martin.*]

SOLUTION.

The distance of the centre of gravity of the bowl from the centre of the rim is

$$\frac{\pi (R^4 - r^4)}{4M},$$

where M is the volume of the bowl.

Let α be the angle of inclination of the bowl to the horizon, and take moments about the point of support, the vertical through which passes through the centre of the rim, the line of action of the upward pressure being normal to the surface.

We get

$$\tan \alpha = \frac{16Rr^3}{3(R^4 - r^4)}. \qquad [Geo.\ R.\ Dean.]$$

EXERCISES.

366

Show that for any parabola $y = x^2 + ax + b$, the area included between the curve, any two ordinates, and the x-axis, is equal to the product of the ordinate midway between the bounding ordinates and the interval between them, plus one-twelfth the cube of this interval. [*W. H. Echols.*]

367

On the axis of x take any point x_m, and equi-distant from it on either side select points a and b. Draw verticals through a and b, also the circle on ab as diameter. With a and b as centers and radius equal to a side of the inscribed square draw arcs cutting the verticals at a and b in a' and b'. The straight lines $a'x_m$ and $b'x_m$ cut the circle in points whose abscissæ are x_1 and x_2, respectively.

Show that if any cubic* of the form

$$y = x^3 + \alpha x^2 + \beta x + \gamma$$

be described passing through any two points x_1y_1 and x_2y_2 on the verticals at a and b respectively, then the area included between the curve, the verticals at a and b, and the x-axis, is constant and equal to

$$\tfrac{1}{2}(y_1 + y_2)(b - a). \qquad [W.\ H.\ Echols.]$$

368

A homogeneous ellipsoid of weight W and semi-axes a, b, c has a weight P suspended from a point on the surface whose coordinates, referred to the axial planes, are l, m, n. Find the inclinations of the axial planes to the horizontal plane upon which the ellipsoid rests. [*H. Y. Benedict.*]

* Of course the same applies to any curve of lower degree, as the parabola

$$y = x^2 + px + q.$$